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Dual interpretations of Seiberg-Witten and Dijkgraaf-Vafa curves

Karl Landsteiner and Sergio Montero

Instituto de Física Teórica, C-XVI Universidad Autónoma de Madrid
28049 Madrid, Spain
E-mail: Karl.Landsteiner@uam.es, Sergio.Montero@uam.es

Abstract: We give dual interpretations of Seiberg-Witten and Dijkgraaf-Vafa (or matrix model) curves in $\mathcal{N} = 1$ supersymmetric $\mathrm{U}(N)$ gauge theory. This duality interchanges the rank of the gauge group with the degree of the superpotential; moreover, the constraint of having at most log-normalizable deformations of the geometry is mapped to a constraint in the number of flavors $N_f < N$ in the dual theory.

Keywords: Duality in Gauge Field Theories, Brane Dynamics in Gauge Theories, Supersymmetry and Duality, Supersymmetric Effective Theories.

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Contents

1. Introduction

Supersymmetric gauge theories are of great interest not only as possible extensions of the standard model of particle physics, but also as a theoretical laboratory to develop and test ideas about the strong coupling dynamics of gauge theories in general. Holomorphicity arguments allow to obtain exact non-perturbative results. In $\mathcal{N} = 1$ supersymmetric gauge theories, the exactly computable holomorphic quantities include the superpotential. Relying on arguments from string theory Dijkgraaf and Vafa argued [1] that the exact superpotential can be computed from the partition function of a simple matrix model; the action of it being given by the tree level superpotential of the gauge theory. The loop equations of the matrix model give rise to an algebraic Riemann surface in the large-N limit and the partition function can be calculated from a set of contour integrals on this Riemann surface. The superpotential can then in turn be computed from the partition function of the matrix model.

Using the special properties of chiral operators in supersymmetric theories, Cachazo et al. showed in [2] that the matrix model results can be obtained directly in gauge theory through a generalization of the Konishi anomaly equations [3]. The Konishi anomaly relations give rise to the same Riemann surface as the matrix model loop equations and so, the superpotential of the gauge theory can be calculated in terms of the same contour integrals that determine the matrix model partition function.

One of the most studied examples in this context is $\mathcal{N} = 1$ U(N) gauge theory with a chiral matter multiplet ϕ in the adjoint representation, N_f chiral multiplets in the fundamental and antifundamental representation and a tree level superpotential of the form

$$W_{\text{tree}} = W(\phi) + \sum_{f=1}^{N_f} \widetilde{Q}^f(\phi + m_f)Q_f\,, \qquad W(\phi) = \sum_{k=0}^{m} \frac{g_k}{k+1}\,\text{tr}(\phi^{k+1})\,. \tag{1.1}$$

Classically this model has a set of discrete vacua characterized by the eigenvalues a_i of the vacuum expectation value of ϕ. Each eigenvalue can be N_i times degenerate, the gauge group is broken to $\prod_{i=1}^m \mathrm{U}(N_i)$ and $\sum_{i=0}^m N_i = N$. The eigenvalues a_i are the solutions of $W'(z) = 0$ and the vacuum expectation values of the fundamentals vanish. In addition there are Higgs vacua with $\langle \widetilde{Q}^f Q_f \rangle \neq 0$ and a corresponding eigenvalue of ϕ, which we call $z_f = -m_f$. In the Higgsed vacua the rank of the gauge group is reduced such that $\sum_{i=1}^m N_i < N$.

Although Higgs and Coulomb vacua are clearly distinguished phases classically, it is well known that in theories with matter transforming faithfully under the center of the gauge group there is no phase distinction between the two [4]. In supersymmetric gauge theories this is also well-established in $\mathcal{N} = 2$ [5] and in the context of generalized Konishi anomalies in [6] and in particular in [7].

The generalized Konishi anomaly relations determine relations for the vacuum expectation values of the generating functions

$$T(v) = \left\langle \frac{1}{v - \phi} \right\rangle , \qquad R(v) = \left\langle -\frac{1}{32\pi^2} \frac{\mathcal{W}^\alpha \mathcal{W}_\alpha}{v - \phi} \right\rangle . \tag{1.2}$$

Introducing $y(v) = 2R(v) + W'(v)$ the Konishi anomaly relation for R takes the form of a hyperelliptic Riemann surface

$$y^2 = (W'(v))^2 - f_{m-1}(v) , \tag{1.3}$$

where f_{m-1} is a polynomial of degree $m-1$ and its coefficients determine the gaugino condensates S_i in the i-th factor group.

The solution for $T(v)$ is

$$T(v) = \sum_{f=1}^{N_f} \left(\frac{1}{2(v+m_f)} + \frac{(1-2r_f) y(q_f)}{2y(v)(v+m_f)} \right) + \frac{c(v)}{y(v)} , \tag{1.4}$$

where q_f is the point on the Riemann surface (1.3) lying above $v = -m_f$ on the upper sheet,[1] r_f is one if the f-th flavor gives rise to higgsing, otherwise $r_f = 0$ and $c(v)$ is a polynomial of degree $m-1$ whose coefficients determine the ranks N_i of the factor groups. The gaugino condensates and the ranks of the factor groups are determined by the compact period integrals on Σ as

$$S_i = \oint_{A_i} R(v)\, dv , \qquad N_i = \oint_{A_i} T(v)\, dv . \tag{1.5}$$

These expressions allow to compute the effective superpotential off-shell as a holomorphic function of the gaugino bilinears S_i. The on-shell superpotential is obtained then by minimizing with respect to the S_i. On-shell, all the compact periods of T are integer valued [7].

[1] The Riemann surface (1.3) is a two-sheeted cover of the v-plane. The sheet on which $y(v) \approx S/v$ for large values of v is called upper sheet.

This demands that the on-shell T is the derivative of the logarithm of a meromorphic function on (1.3). It can be written as

$$T(v) = \frac{\partial}{\partial v} \log \left(P(v) + \sqrt{P^2 - 4\Lambda^{2N-N_f} \prod_{f=1}^{N_f} (v + m_f)} \right) , \tag{1.6}$$

where $P(v)$ is a polynomial of degree N. The condition that $T(v)$ is a meromorphic differential on Σ implies the factorization

$$y_{\mathrm{SW}}^2 = P_N^2(v) - 4\Lambda^{2N-N_f} \prod_{f=1}^{N_f} (v + m_f) = Q_{2n}(v) S_{N-n}(v)^2 , \tag{1.7}$$

$$y_{m.m.}^2 = (W_m'(v))^2 - f_{m-1}(v) = Q_{2n}(v) H_{m-n}(v)^2 , \tag{1.8}$$

where F, H and Q are polynomials of the degree indicated as an index. It is furthermore obvious that the first equation is the Seiberg-Witten [12] curve of the $\mathcal{N} = 2$ gauge theory that is obtained by setting the tree level couplings g_k to zero. From the standpoint of Seiberg-Witten theory the factorization reflects the fact that addition of a superpotential in ϕ localizes the Coulomb branch of the $\mathcal{N} = 2$ moduli space to the subloci where n monopoles (or dyons) become massless. These are precisely the loci where the Seiberg-Witten curve develops n double points. A particular solution to this factorization condition (1.7) represents a choice of vacuum in the gauge theory. Moreover, the solutions to the factorization conditions include vacua on the pseudo-confining as well as the pseudo-higgs branch, and both branches can be connected by varying the masses such that q_f, the image point of m_f on (1.3) crosses a branch cut [7, 8].

These factorization conditions have been given a beautiful interpretation in [9]. The gauge theory is realized on a single M-theory fivebrane whose worldvolume is partially wrapped on a Riemann surface embedded in a three dimensional complex space. The Riemann surface is described by two algebraic equations. Each of them can be seen as a projection of the Riemann surface onto a two dimensional subspace. To represent the ground state of the gauge theory the two Riemann surfaces have to be proportional to each other, which means that they have to factorize onto a common reduced Riemann surface just as in (1.7), i.e. the Riemann surface on which the M-theory fivebrane is wrapped is given precisely by equations (1.7). We point out that there are two different $\mathcal{N} = 1$ gauge theories with different rank and superpotential who produce the same projections. This observation is the starting point for our argument. We will see that the same set of equations and factorization conditions can be obtained from two different gauge theories. What in one gauge theory represents the Seiberg-Witten curve is interpreted as the matrix model curve in the other gauge theory and vice versa. Thus we find pairs of "dual" gauge theories which in some sense share the same ground state. A somewhat surprising feature of this "duality" is that the rank of the first gauge group determines the degree of the tree level superpotential of the other gauge theory.

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This paper is organized as follows. In section 2 we review first the M-theory fivebrane construction of $\mathcal{N}=1$ supersymmetric gauge theories and show how the same curve emerges from two different gauge theories.

In section 3 we compute the exact superpotential of the "dual" gauge theories. In order to do so we first prove that the exact superpotential as computed from the geometry of the Riemann surface coincides with the low energy field theoretical superpotential. Here we extend the known proofs in the literature to the case of arbitrary high degree superpotential and matter content. We then compute explicitly the superpotentials in two simple cases.

Section 4 discusses the results obtained and offers some conclusions.

2. The M-theory fivebrane

One of the most useful tools to model non-perturbative dynamics of supersymmetric gauge theories is the M-theory fivebrane. In particular, four dimensional gauge theories can be constructed on the worldvolume of configurations of intersecting D- and NS-branes in type-IIA string theory in the weak coupling limit. The strong coupling dynamics is modelled by going to the strong coupling M-theory limit of IIA string theory.

The best known and studied example is $\mathrm{SU}(N)$ supersymmetric gauge theory, realized on the worldvolume of D4-branes that are spanned between two NS5-branes. Two different situations can be distinguished: one with parallel fivebranes and resulting in $\mathcal{N}=2$, the other one with non-parallel fivebranes resulting in $\mathcal{N}=1$. In the following we will review the construction in detail and also recall the beautiful interpretation as a unification of DV (Dijkgraaf-Vafa) and SW (Seiberg-Witten) curves as pointed out in [9].

Let us start in type-IIA string theory. The NS-fivebranes will span the directions $x^0,\ldots,x^3,x^4,x^5$ and are separated by some distance, say L, in the x^6 direction. Otherwise they are parallel. In between these fivebranes we span N D4-branes along the directions $x^0,\ldots,x^3,x^6$. In the x^6 direction the D4-branes terminate on the fivebranes. This brane construction engineers an $\mathcal{N}=2$ supersymmetric gauge theory with gauge group $\mathrm{SU}(N)$ on the non-compact, $3+1$ dimensional part of the worldvolume of the D4-branes [10]. So far we have constructed pure gauge theory. To include also matter (hyper)multiplets in the fundamental representation of $\mathrm{SU}(N)$ there are two ways to achieve this. One way is to use D6-branes spanning the $x^0,\ldots,x^3,x^7,x^8,x^9$ directions and located in between the NS-branes in the x^6 direction. An alternative is to consider semi-infinite D4-branes ending on only one of the NS-branes, e.g. D4-branes spanning all of x^6 (and $x^0,\ldots,x^3$) to the right of the rightmost NS-brane. Both possibilities are equivalent through the so called Hanany-Witten transition [11]. We will however focus in this paper on the realization of fundamental matter through D4-branes only.

This type-IIA brane configuration can be lifted to M-theory and represents then the strong-coupling dynamics of the gauge theory. In M-theory, the D4-branes are lifted to M-theory fivebranes wrapped on the additional circle and what appears as intersecting branes in IIA string theory is lifted to a single M-theory fivebrane wrapping a complex curve. We introduce complex coordinates $v=x^4+ix^5$ and $t=\exp(-(x^6+ix^{10})/R)$, where x^{10} parameterizes the eleventh dimension of M-theory, it is supposed to be compact with identification $x^{10} \equiv x^{10} + 2\pi R$. Type-IIA string theory is recovered in the limit $R \to 0$. The M-theory fivebrane is partially wrapped on the Riemann surface

$$t^2 - 2P_N(v)t + 4\Lambda^{2N-N_f} \prod_{f=1}^{N_f} (v + m_f) = 0\,, \tag{2.1}$$

Here $P_N(v)$ is a polynomial of degree N in v. In the classical or type-IIA limit the zeroes of $P_N(v)$ correspond to the locations of the D4-branes in the v-plane. Similarly m_f are the positions of the semi-infinite fourbranes to the right and represent the masses of the fundamental hypermultiplets in the gauge theory. Of course, the Riemann surface (2.1) is the Seiberg-Witten curve of the corresponding $\mathcal{N} = 2$ supersymmetric gauge theory with N_f hypermultiplets in the fundamental representation. An additional ingredient in Seiberg-Witten theory is the differential $\lambda_{\mathrm{SW}} = v\frac{dt}{t}$. Its period integrals around the cuts of (2.1) define special coordinates a_i and the prepotential $\mathcal{F}(a_i)$ as

$$a_i = \frac{1}{2\pi i} \oint_{A_i} \lambda_{\mathrm{SW}}\,, \qquad \frac{\partial \mathcal{F}}{\partial a_i} = \frac{1}{2\pi i} \oint_{B_i} \lambda_{\mathrm{SW}}\,. \tag{2.2}$$

Here A_i, B_i form a symplectic basis of (compact) one-cycles. In the classical limit the a_i's can be interpreted as the eigenvalues of the expectation value of the adjoint field $\langle\phi\rangle$ forming part of the $\mathcal{N} = 2$ vector multiplet. We find therefore

$$\lambda_{\mathrm{SW},\,cl.} = v\,\mathrm{tr}\left(\frac{dv}{v - \phi}\right), \tag{2.3}$$

since this differential has poles with residue a_i at $z = a_i$ and therefore is the classical limit of the Seiberg-Witten differential. This suggests to interpret

$$\left\langle \mathrm{tr}\left(\frac{dv}{v-\phi}\right)\right\rangle = T(v)\,dv = \frac{dt}{t}\,. \tag{2.4}$$

in the full quantum theory. That the differential T is given by (2.4) has been already obtained in [9]. We just comment that this differential is already well defined in the $\mathcal{N} = 2$ theory and that it has the simple relation to the Seiberg-Witten differential $v\,T(v)\,dv = \lambda_{\mathrm{SW}}$.

Now we want to break to $\mathcal{N} = 1$ by adding a tree level superpotential for the adjoint chiral multiplet

$$W(\phi) = \sum_{k=0}^{m} \frac{g_k}{k+1}\,\mathrm{tr}(\phi^{k+1})\,. \tag{2.5}$$

In the type-IIA brane configuration this is realized by taking m NS-fivebranes instead of only one on the right side (NS' branes). We introduce the complex coordinate $w = x^7 + ix^8$. The worldvolume of the NS'-fivebranes have to span the lines $w = v - \varphi_i$ where φ_i are the roots of $W'_{\mathrm{tree}}(x) = \sum_{k=0}^{m} g_k x^k = g_m \prod_{i=1}^{m}(x - \varphi_i) = 0$. In the following we will set the highest coupling g_m — which determines the degree of the superpotential — to $g_m = 1$.

The M-theory lift of $\mathcal{N}=1$ brane configuration has been obtained in [13–15]. In the M-theory fivebrane realization, large v values correspond to semiclassical physics. In order to implement the superpotential (1.1) in the M-theory fivebrane configuration we have to impose the asymptotic boundary conditions [15]

$$v \to \infty \begin{cases} w \to 0\,, & t \sim v^N \\ w \to W'(v)\,, & t \sim 4\Lambda^{2N-N_f} v^{N_f-N}\,. \end{cases} \tag{2.6}$$

From these boundary conditions a holomorphic function $w(t,v)$ on the Riemann surface (2.1) can be constructed. The two assumptions that allow to find w are that the $\mathcal{N}=2$ Seiberg-Witten curve remains unchanged[2] and that $w(t,v)$ is a rational function with no additional poles at finite values of v. These allow to constrain w to the form

$$w = N(v) + \frac{H(v)}{S(v)}\Big(t - P_N(v)\Big)\,. \tag{2.7}$$

We assumed here that potentially present double points in the Seiberg-Witten curve (2.1) are collected in the polynomial $S(v)$

$$P_N^2 - 4\Lambda^{2N-N_f}\prod_{f=1}^{N_f}(v+m_f) = S_{N-n}(v)^2\, Q_{2n}(v)\,, \tag{2.8}$$

where $Q(v)$ is a polynomial of degree $2n$ and $S(v)$ a polynomial of degree $N-n$. Let us momentarily denote the two branches of t as $t_\pm = P_N \pm \sqrt{P_N^2 - 4\Lambda^{2N-N_f}\prod_{f=1}^{N_f}(v+m_f)}$. We find then

$$w_\pm(t_\pm, v) = N(v) \pm H(v)\sqrt{Q(v)}\,. \tag{2.9}$$

The asymptotic boundary conditions fix $N(v)$ to be

$$N(v) = [H(v)\sqrt{Q(v)}]_+ = W'(v)\,, \tag{2.10}$$

and finally one finds that w fulfills a quadratic equation

$$w^2 - 2W'_m(v)w + f_{m-1}(v) = 0\,. \tag{2.11}$$

where $f_{m-1}(v)$ is a polynomial of degree $m-1$ which can also be written as $f_{m-1} = N(v)^2 - H(v)^2 Q(v)$.

Note. Of course there is nothing new in the above discussion except for one minor point: contrary to the original work in [15] we do not assume that the degree $m+1$ of the superpotential $W(v)$ is limited by $m < N$. We also allowed in the above discussion arbitrarily high values of m, and as one can convince oneself easily nothing changes in the discussion compared to [15]. To limit oneself to add superpotentials of degree up to $m < N$ is of

[2] This fact can be seen in a naïve way. Breaking to $\mathcal{N}=1$ means we rotate the NS'-branes in the (w,v)-plane. However, looking from the (t,v)-plane, which is how the construction in [10] is regarded, one sees no change in the type-IIA picture. In this way, one effectively sees the curve does not change.

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course motivated by the fact that in field theory operators of the form $\mathrm{tr}(\phi^{N+k})$ are not independent from the operators $\mathrm{tr}(\phi^k)$ with $k \in 1 \ldots N$. They can be expressed as products of the latter ones in the form of multi-trace operators. Quantum mechanically these expressions get modified, and we will consider the higher power single trace operators as basic objects in the theory. This is also the standard way of treating these operators in Dijkgraaf-Vafa theory.[3]

The $\mathcal{N}=1$ supersymmetric gauge theory with N_f hypermultiplets in the fundamental representation and superpotential $W_{\mathrm{tree}}(\phi)$ is therefore described in M-theory by a Riemann surface. It is embedded in a three-dimensional complex space labelled by v, t, w and defined by the two equations

$$t^2 - 2P_N(v)t + 4\Lambda^{2N-N_f}\prod_{f=1}^{N_f}(v+m_f) = 0\,, \tag{2.12}$$

$$w^2 - 2W'_m(v)w + f_{m-1}(v) = 0\,. \tag{2.13}$$

Furthermore the asymptotic boundary conditions impose the factorization conditions

$$P_N(v)^2 - 4\Lambda^{2N-N_f}\prod_{f=1}^{N_f}(v+m_f) = S_{N-n}(v)^2\, Q_{2n}(v)\,, \tag{2.14}$$

$$W'_m(v)^2 - f_{m-1}(v) = H_{m-n}(v)^2\, Q_{2n}(v)\,. \tag{2.15}$$

Equation (2.13) has precisely the form of the matrix model curve in Dijkgraaf-Vafa theory, whereas (2.12) has the form of Seiberg-Witten curve as said previously. The gauge theory and matrix model resolvents are given by

$$\left\langle \mathrm{tr}\left(\frac{dv}{v-\phi}\right)\right\rangle = T(v)\,dv = \frac{dt}{t}\,, \tag{2.16}$$

$$\left\langle -\frac{1}{16\pi^2}\,\mathrm{tr}\left(\frac{\mathcal{W}_\alpha \mathcal{W}^\alpha}{v-\phi}\right)dv\right\rangle = 2R(v)\,dv = w dv\,. \tag{2.17}$$

The effect of the factorizations is to modify the Riemann surface one has in the $\mathcal{N}=2$ theory. After adding the tree level superpotential, the moduli space is lifted and we are restricted to some singular submanifolds in it. The remaining cycles of the Riemann surface are given by the single roots associated to the $Q_{2n}(v)$ polynomial.

2.1 Suggestion of the duality

As seen, in the above construction we have obtained the two equations (2.12) and (2.13) defining our M-theory fivebrane. We got also two factorizations for them, (2.14) and (2.15), which tell us where the singular points for that Riemann surface are. These two curves, (2.12) and (2.13), have been traditionally regarded as the Seiberg-Witten and Dijkgraaf-Vafa curves, respectively. This is motivated by the fact that one *first* constructs the $\mathcal{N}=2$ curve, i.e. the SW curve, and *afterwards* performs the brane rotations to get $\mathcal{N}=1$, i.e.

[3] See however [16] for the case of multi-trace operators.

the DV curve [1]. Let us go back for a moment to equation (2.7) and take into account the asymptotic boundary contition (2.6), which is $w = W'_m(v) + \frac{H(v)}{S(v)}(t - P_N(v))$. This can be trivially inverted to find t as a meromorphic function on the Riemann surface (2.13)

$$t = P_N(v) + \frac{S(v)}{H(v)}\Big(w - W'_m(v)\Big)\,. \tag{2.18}$$

In fact t regarded as a function on the Riemann surface (2.13) has precisely the same properties as w as a function on (2.12): it is rational and does not have any poles at finite values of v. Note that this is the decisive condition which uniquely fixes the form of the Riemann surface (2.13) once the asymptotic boundary conditions are supplied. But it also automatically gives t as a rational function on this Riemann surface with the asymptotic behavior

$$v \to \infty \begin{cases} t \to 0\,, & w \sim v^m \\ t \to P_N(v)\,, & w \sim 0\,. \end{cases} \tag{2.19}$$

These boundary conditions are consistent with the ones in (2.6) as long as $N_f < N$. This implies that the factorization conditions (2.14) and (2.15) could have been obtained by starting with the curve (2.13) and then construct t as a rational function without poles at finite values of v on it and demanding the boundary conditions (2.19). In such an approach (2.13) would be interpreted as the Seiberg-Witten curve of a gauge theory with U(m) gauge group and $\widetilde{N}_f$ fundamental flavors whose masses are given by writing

$$f_{m-1} = 4\widetilde{\Lambda}^{2m-\widetilde{N}_f}(v + \widetilde{m}_f)\,, \tag{2.20}$$

where the number of flavors is at most $\widetilde{N}_f = m - 1$. We see that there is a second "dual" $\mathcal{N} = 1$ supersymmetric gauge theory which we can associate with the system of algebraic curves (2.12), (2.13) thus giving rise to two "dual" interpretations. Let us list the most striking features of this duality:

1. We are interchanging the two polynomials P_N and W'_m, so we have a duality between two supersymmetric gauge theories with different gauge groups and matter content. One is a theory with gauge group $G = \mathrm{U}(N)$ and N_f matter hypermultiplets, whereas the other is a theory with gauge group U(m) and up to $\widetilde{N}_f = m-1$ hypermultiplets. Of course, this implies that a meaningful dual interpretation as gauge dynamics exists only for $m > 1$, i.e. we need at least cubic superpotentials!

2. In principle we can put as many NS' branes as we want. We might write the number of NS' branes as $m = N - k$, where k ranges from all negative integers up to $N - 1$ (since at least we would like to have one NS' fivebrane.) Now, the highest term in the superpotential goes like ϕ^{m+1}; therefore if we do not want powers in the superpotential higher than N — the rank of the gauge group — we must keep $k > 0$. However, with this values set, in the dual interpretation we get

$$\begin{aligned} \text{rank}\,\widetilde{G} &= m \\ \deg \widetilde{W} &= m + k + 1 \quad \text{(remember } k > 0 \text{ here.)} \end{aligned}$$

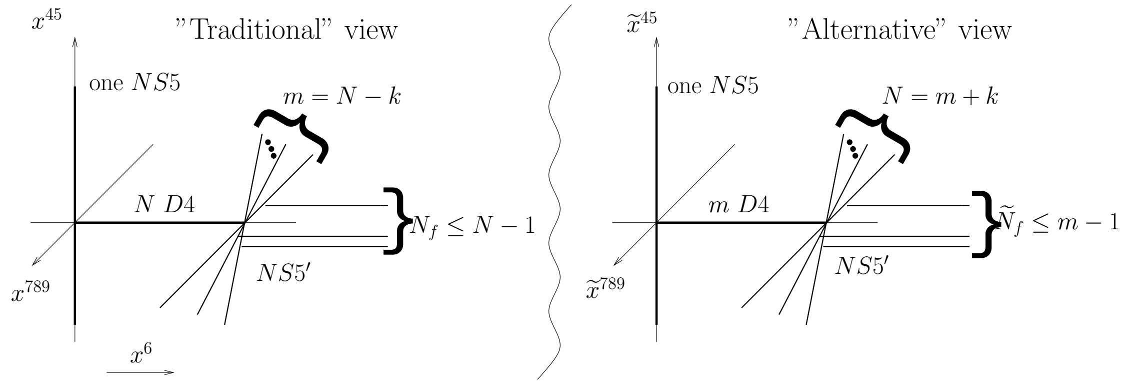

Figure 1: A figure with the two possible (dual) descriptions of the same $\mathcal{N} = 1$ equations defining the M-theory fivebrane.

and thus we *necessarily* generate higher order terms for the dual superpotential.

One could do it all the way around, with $k \leq 0$. Then one has higher order terms in the traditional theory and lower ones in the dual.

3. The argument above supports the idea of **duality** in this construction. If in one picture we have lower order operators, in the other we irremediably have higher order terms, and vice versa. Thus, applying twice the map we come back to our original theory.

4. Note that one always has a *constraint* in the matter content of the form $N_f \leq N - 1$ if the duality is to exist. This comes from consistency of the asymptotic boundary conditions in the dual interpretations or more simply from the fact that in the matrix model curve the degree of f_{m-1} is determined through the condition that it represents normalizable or at most log-normalizable deformations! Therefore the constraint of allowing at most log-normalizable deformations is mapped through our duality to $N_f \leq N - 1$. As it is well known, gauge theories with $N_f \geq N$ are indeed qualitatively different from the one with $N_f < N$, e.g. complete higgsing of the gauge group is possible, baryonic operators can be formed and in the massless case moduli spaces exist instead of runaway vacua.

5. The relevant differentials entering the computation of gaugino condensates and ranks of the factor groups are

$$T(v)\, dv = \frac{dt}{t} \qquad 2R(v) = w\, dv \text{ for the U}(N) \text{ gauge theory} \tag{2.21}$$

$$\widetilde{T}(v)\, dv = \frac{dw}{w} \qquad 2\widetilde{R}(v) = t\, dv \text{ for the U}(m) \text{ gauge theory}\,. \tag{2.22}$$

6. Notice that the M-theory fivebrane construction does not physically give rise to this duality. In the M-theory setup the coordinate t is singled out by the holomorphic threeform $\Omega = \frac{dt}{t} \wedge dw \wedge dv$, which determines the volume form $\Omega \wedge \overline{\Omega}$ and determines $\mathrm{Im}(\log(t))$ to be compact. The dual theory would demand the threeform to be $\widetilde{\Omega} =$

$dt \wedge \frac{dw}{w} \wedge dv$ and therefore indicating that $\mathrm{Im}(\log(w))$ is compact. Of course the original M-theory/type-IIA background is not compact in this direction. It has to be emphasized, however, that this information is not needed to compute the effective superpotential. All one needs is the embedding (2.12), (2.13) of the Riemann surface in the complex three dimensional space parameterized by t, w, v and the differentials (2.21), (2.22). In fact, as explicitly demonstrated in [9], the M-theory fivebrane does not reproduce the gauge theory physics correctly beyond that, e.g. Kähler terms computed from the M-theory fivebrane do not allow a gauge theory interpretation. So there is no reason to associate the M-fivebrane with the gauge theory beyond the information that can be obtained from the embedding of the Riemann surface in the three dimensional complex space and some choice of differentials T and R or $\widetilde{T}$ and $\widetilde{R}$. One might therefore interpret the space spanned by t, v, w as an abstract space and forget about the M-theory origin of the construction.

3. Exact "dual" superpotentials

We want to compute explicitly now in some examples the exact superpotentials of the "dual" gauge theories. We will do this in a strong coupling approach, viewing the tree level superpotential for the adjoint scalar as a perturbation to the $\mathcal{N} = 2$ theory. We follow the work in [17], where the equivalence of the superpotential computed from the geometry with the one computed from a low energy field theoretical approach has been proved for the $\mathrm{U}(N)$ super Yang-Mills case. This has been extended later in [18] to the case with flavors with $N_f < N$ and in [8] for the case with $N \leq N_f < 2N$.

However both [18] and [8] considered superpotentials of at most degree N, i.e. the highest adding terms of the form $\mathrm{tr}(\phi^k)$ with $k_{\max} = m + 1 \leq N$. As we have seen before, our duality forces us to consider the cases with $k_{\max} > N$ as well, because if the original theory has a superpotential of degree less than or equal to N, the dual will be a theory with a superpotential of degree $\widetilde{m} + 1 \leq N + 1$ and gauge group $\mathrm{U}(\widetilde{N})$, where $\widetilde{N} = m$. The operators with $k > N$ (or $\widetilde{k} > \widetilde{N}$) are of course not independent from the ones with lower powers and can be expressed as function of the latter. This has to be imposed as additional constraints on the low energy superpotential as in [19].

We therefore want to prove in the following the equivalence of the low energy field theory superpotential, denoted by W_{low} with the superpotential computed from geometry denoted by W_{eff}, in the case with general superpotential and $N_f < 2N$.[4] Following [17], we have to show

$$W_{\text{low}}(g_k, \Lambda, m_f)\Big|_{\Lambda\to 0} = W_{\text{eff}}(\langle S_i\rangle)\Big|_{\Lambda\to 0}, \tag{3.1}$$

$$\frac{\partial W_{\text{low}}(g_k, \Lambda, m_f)}{\partial \log \Lambda^{b_0}} = \frac{\partial W_{\text{eff}}(\langle S_i\rangle)}{\partial \log \Lambda^{b_0}}, \tag{3.2}$$

where $b_0 = 2N - N_f$ is the one-loop beta-function coefficient for $\mathrm{U}(N)$ gauge group.

[4] Even though for the duality to hold we will always have $N_f < N$, the rank of the gauge group.

3.1 Field theory superpotential

We first show that the matrix model curve follows from the field theory superpotential.

The moduli space for the Coulomb branch of the $\mathcal{N}=2$ theory is N dimensional. Once we break to $\mathcal{N}=1$ by adding the tree level superpotential (1.1), its effect is to lift the moduli space except for some singular submanifolds, where $N-n$ massless dyons appear. We parameterize the moduli space by $U_k = \frac{1}{k}\langle \mathrm{tr}(\phi^k)\rangle$.

The polynomial entering the Seiberg-Witten curve is defined as

$$P_N(v) = \langle \det(v-\phi)\rangle \,,$$

which using the solution for $T(v)$ can be written as [7]

$$P_N(v) = v^N \exp\left(-\sum_{r=1}^{\infty}\frac{U_r}{v^r}\right) + \frac{F_{N_f}(v)}{4v^N}\exp\left(+\sum_{r=1}^{\infty}\frac{U_r}{v^r}\right)\,, \tag{3.3}$$

where we denoted $F_{N_f} = 4\Lambda^{b_0}\prod_{f=1}^{N_f}(v+m_f)$. The negative powers in v give the quantum corrected relations between the U_k with $k>N$ and the independent single trace operators with $k \leq N$.

The low energy superpotential is given by[5]

$$\begin{aligned} W_{\text{eff}} \; &= \sum_{k=0}^{m} g_k U_{k+1} + \oint V_{m-N}(x)\left(x^N e^{-\sum_{r=1}^{\infty}\frac{U_r}{x^r}} + \frac{F_{N_f}(x)}{4x^N}\, e^{+\sum_{r=1}^{\infty}\frac{U_r}{x^r}}\right) dx \, + \\ &+ \sum_{i=1}^{N-n}\left(L_i\oint\frac{P_N(x)+\epsilon_i\sqrt{F_{N_f}(x)}}{x-p_i}dx + B_i\oint\frac{P_N(x)+\epsilon_i\sqrt{F_{N_f}(x)}}{(x-p_i)^2}dx\right). \end{aligned} \tag{3.4}$$

$V_{m-N}(x)$ is a polynomial of order $m-N$ whose coefficients are the Lagrange multipliers enforcing the relations for the higher power U_k's. On the singular submanifold the Seiberg-Witten curve has $N-n$ double points p_i. These double points are enforced by the Lagrange multipliers L_i and B_i, furthermore $\epsilon_i = \pm 1$. The equations of motion for B_i and p_i tell us that, if we do not want to have triple or higher roots, then $B_i = 0$ on-shell. Differentiating with respect to U_{r+1} yields

$$\frac{\partial W_{\text{eff}}}{\partial U_{r+1}} = g_r + \oint V_{m-N}(x)\frac{\partial P_N(x)}{\partial U_{r+1}}\,dx + \sum_{i=1}^{N-n}\oint \frac{L_i}{x-p_i}\frac{\partial P_N(x)}{\partial U_{r+1}}\,dx = 0\,, \tag{3.5}$$

where using (3.3) we obtain

$$\frac{\partial P_N(x)}{\partial U_{r+1}} = \frac{1}{x^{r+1}}\left[P_N(x) - 2x^N\exp\left(-\sum_{i=1}^{\infty}\frac{U_i}{x^i}\right)\right]_+ = -\left[\frac{\sqrt{P_N^2 - F_{N_f}}}{x^{r+1}}\right]_+ . \tag{3.6}$$

[5]There is an implicit $(2\pi i)^{-1}$ in the definition of $\oint$.

Thus we have an expression for the g_k's, which we can multiply by v^k and sum over k to obtain the derivative of the tree level superpotential

$$W'(v) = \sum_{k=0}^{\infty}\left(\oint V_{m-N}(x) + \sum_{i=1}^{N-n}\oint \frac{L_i}{x-p_i}\right)\frac{v^k}{x^{k+1}}\sqrt{P_N(x)^2 - F_{N_f}(x)}\,dx\,, \tag{3.7}$$

where we have extended the summation to infinity since the terms for $k > m$ do not contribute. We write

$$\sum_{i=1}^{N-n}\frac{L_i}{x-p_i} = \frac{\widehat{H}_{N-n-1}(x)}{S_{N-n}(x)}\,, \tag{3.8}$$

and use the factorization of the SW curve to obtain

$$W'(v) = \oint \frac{y_{m.m.}(x)}{x-v}\,dx\,, \tag{3.9}$$

where the integral is over a large contour at infinity and

$$\begin{aligned} y^2_{m.m.} &= H^2_{m-n}(v)\,Q_{2n}(v) = W'_m(v)^2 + \mathcal{O}(v^{m-1})\,, \\ \text{with } H_{m-n} &= V_{m-N}S_{N-n} + \widehat{H}_{N-n-1}\,. \end{aligned} \tag{3.10}$$

This gives a strong coupling field theory derivation of the matrix model curve in the case valid for $N_f < 2N$ and arbitrary degree superpotential.

Since the breaking pattern is $\mathrm{U}(N) \to \prod_{i=1}^{n}\mathrm{U}(N_i)$, the classical limit of the superpotential is

$$W_{\text{low,classical}} = \sum_{i=1}^{n} N_i W(\varphi_i)\,. \tag{3.11}$$

We now compute the dependence of the low energy superpotential on the scale Λ^{b_0}:

$$\begin{aligned} \frac{\partial W_{\text{low}}(g_k,\Lambda)}{\partial \log\Lambda^{b_0}} &= \oint_{\mathcal{C}_{\text{large}}} V_{m-N}(x)\frac{F_{N_f}(x)}{4x^N}\;e^{+\sum_{r=1}^{\infty}\frac{U_r}{x^r}}\,dx + \sum_{i=1}^{N-n}\frac{L_i\,\epsilon_i}{2}\sqrt{F_{N_f}(p_i)} \\ &\quad + \oint_{\mathcal{C}_{\text{large}}}\sum_{i=1}^{N-n}\frac{L_i}{x-p_i}\frac{F_{N_f}(x)}{4x^N}e^{+\sum_{r=1}^{\infty}\frac{U_r}{x^r}}\,dx\,. \end{aligned} \tag{3.12}$$

From the geometry it follows (as we will review shortly in the next section) that the logarithmic derivative with respect to Λ^{b_0} is given by the highest coefficient b_{m-1} of the polynomial $f_{m-1} = \sum_{i=0}^{m-1} b_i\,v^i$.

In order to compute b_{m-1} we note that[6]

$$W'_m(v) - \sqrt{W'_m(v)^2 - f_{m-1}(v)} = \frac{b_{m-1}}{2}\frac{1}{v} + \mathcal{O}\left(\frac{1}{v^2}\right). \tag{3.13}$$

Using both the factorizations of the Dijkgraaf-Vafa and the Seiberg-Witten curves, we can write

$$\begin{aligned} \sqrt{W'_m(v)^2 - f_{m-1}(v)} &= H_{m-n}\sqrt{Q_{2n}} = (V_{m-N}S_{N-n} + \widehat{H}_{N-n-1})\sqrt{Q_{2n}} \\ &= \left(V_{m-N}(v) + \sum_{i=1}^{N-n}\frac{L_i}{v-p_i}\right)\sqrt{P_N(v)^2 - F_{N_f}(v)}\,. \end{aligned} \tag{3.14}$$

[6]Remember that we have set the coupling $g_m = 1$.

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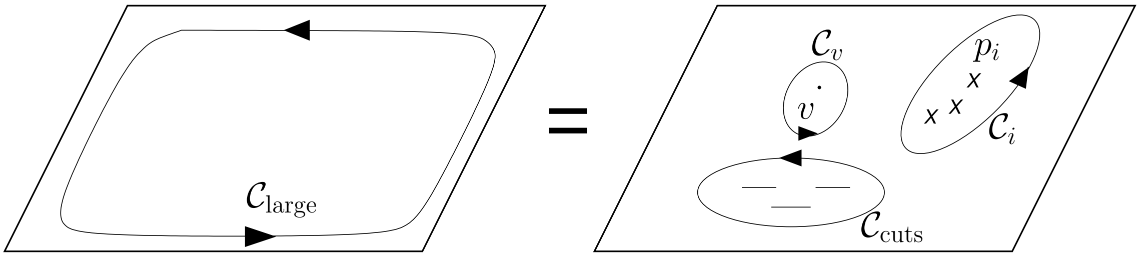

Figure 2: The contour $\mathcal{C}_{\rm large}$ can be deformed into a contour centered at v and a contour $\mathcal{C}_{\rm cuts}$ around the cuts of the Riemann surface. Since the integrand is regular at p_i the contour $\mathcal{C}_i$ does not contribute.

As we have just seen $W'(v)$ can be written as

$$W'(v) = \oint_{\mathcal{C}_{\rm large}} \left(V_{m-N}(x) + \sum_{i=1}^{N-n} \frac{L_i}{x-p_i} \right) \frac{\sqrt{P_N(x)^2 - F_{N_f}(x)}}{x-v}\, dx\,. \tag{3.15}$$

We can deform the large contour $\mathcal{C}_{\rm large}$ into a contour $\mathcal{C}_v$, encircling just the point $x = v$ and $\mathcal{C}_{\rm cuts}$, which encloses the branch cuts of the Riemann surface (see figure 2). Notice that the integrand is regular at $x = p_i$ due to the presence of the double points inside the square root. The contour surrounding $x = v$ yields the same expression as (3.14), so using equation (3.13) we find

$$\frac{b_{m-1}}{2}\frac{1}{v} + \mathcal{O}(v^{-2}) = \oint_{\mathcal{C}_{\rm cuts}} \left(V_{m-N}(x) + \sum_{i=1}^{N-n} \frac{L_i}{x-p_i} \right) \frac{\sqrt{P_N(x)^2 - F_{N_f}(x)}}{x-v}\, dx \tag{3.16}$$

$$\begin{aligned} &= -\frac{1}{v} \oint_{\mathcal{C}_{\rm cuts}} \left(V_{m-N}(x) + \sum_{i=1}^{N-n} \frac{L_i}{x-p_i} \right) \times \\ &\quad \times \sqrt{P_N(x)^2 - F_{N_f}(x)}\, dx + \mathcal{O}(v^{-2})\,, \end{aligned} \tag{3.17}$$

where in the second line we have considered $v \gg 1$. To match this with (3.12) we still have to do some manipulations. We notice that

$$\sqrt{P_N^2 - F_{N_f}} - P_N = -\frac{F_{N_f}}{2x^N}\, e^{+\sum \frac{U_r}{x^r}}\,. \tag{3.18}$$

For the first term inside the brackets of (3.17) we note that we can subtract a $V_{m-N}P_N$ term from the integrand since it is analytic on the whole plane. Thus

$$\begin{aligned} \oint_{\mathcal{C}_{\rm cuts}} V_{m-N}\sqrt{P_N^2 - F_{N_f}}\,dx &= \oint_{\mathcal{C}_{\rm cuts}} V_{m-N}\left(\sqrt{P_N^2 - F_{N_f}} - P_N\right) dx \\ &= \oint_{\mathcal{C}_{\rm large}} V_{m-N}\left(\sqrt{P_N^2 - F_{N_f}} - P_N\right) dx \\ &= -\oint_{\mathcal{C}_{\rm large}} V_{m-N}\,\frac{F_{N_f}}{2x^N}\, e^{+\sum \frac{U_r}{x^r}}\, dx\,, \end{aligned} \tag{3.19}$$

where the last contour deformation could be performed without any problem.

Now we consider the second term in the brackets of (3.17).

$$
\begin{aligned}
\oint_{\mathcal{C}_{\text{cuts}}} \sum_{i=1}^{N-n} \frac{L_i}{x-p_i} \sqrt{P_N^2 - F_{N_f}}\, dx &= \oint_{\mathcal{C}_{\text{cuts}}} \sum_{i=1}^{N-n} \frac{L_i}{x-p_i} \left(\sqrt{P_N^2 - F_{N_f}} - P_N \right) dx \\
&= \left(\oint_{\mathcal{C}_{\text{large}}} - \oint_{\mathcal{C}_i} \right) \sum_{i=1}^{N-n} \frac{L_i}{x-p_i} \left(\sqrt{P_N^2 - F_{N_f}} - P_N \right) dx \\
&= -\oint_{\mathcal{C}_{\text{large}}} \sum_{i=1}^{N-n} \frac{L_i}{x-p_i} \frac{F_{N_f}}{2x^N} e^{+\sum \frac{U_r}{x^r}} dx - \\
&\quad - \oint_{\mathcal{C}_i} \sum_{i=1}^{N-n} \frac{L_i}{x-p_i} \left(\sqrt{P_N^2 - F_{N_f}} - P_N \right) dx \\
&= -\oint_{\mathcal{C}_{\text{large}}} \sum_{i=1}^{N-n} \frac{L_i}{x-p_i} \frac{F_{N_f}}{2x^N} e^{+\sum \frac{U_r}{x^r}} dx - \\
&\quad - \sum_{i=1}^{N-n} L_i \left(\sqrt{P_N(p_i)^2 - F_{N_f}(p_i)} - P_N(p_i) \right). \qquad (3.20)
\end{aligned}
$$

For the last term, we note that $\sqrt{P_N(p_i)^2 - F_{N_f}(p_i)} = 0$, and also $P_N(p_i) = -\epsilon_i \sqrt{F_{N_f}(p_i)}$. Therefore this takes us to the end of the computation and we find

$$
\begin{aligned}
\frac{b_{m-1}}{2} &= \oint_{\mathcal{C}_{\text{large}}} V_{m-N} \frac{F_{N_f}}{2x^N} e^{+\sum \frac{U_r}{x^r}} dx + \oint_{\mathcal{C}_{\text{large}}} \sum_{i=1}^{N-n} \frac{L_i}{x-p_i} \frac{F_{N_f}}{2x^N} e^{+\sum \frac{U_r}{x^r}} dx + \\
&\quad + \sum_{i=1}^{N-n} L_i \epsilon_i \sqrt{F_{N_f}(p_i)}\,. \qquad (3.21)
\end{aligned}
$$

Comparison with (3.12) shows that

$$
\frac{\partial W_{\text{low}}(g_k, \Lambda)}{\partial \log \Lambda^{b_0}} = \frac{b_{m-1}}{4 g_m}\,, \qquad (3.22)
$$

for a number of flavors $N_f < 2N$ and arbitrary degree of the tree level superpotential. For the sake of completeness we also have reinstated the coupling g_m.

3.2 "Dual" geometric superpotential

In the preceding subsection we have identified the expressions for W_{low} in the classical limit and its derivative with respect to $\log \Lambda^{b_0}$. Now we do the same using the Riemann surface. If at low energy we have $\mathrm{U}(1)^n$ "photons", some of our branch cuts will be closed. Since the degree of the superpotential is $m \geq N$, $m-n$ cuts will be closed. From [8] we should assume some $S_i = 0$ from the beginning, implemented by hand off-shell.[7] We will choose

[7] The question when and if $S_i = 0$ implies a closed cut has been thoroughly investigated in [20]. It turned out to be especially important in the case of theories with symplectic or orthogonal gauge group [21–24] where non vanishing cuts for $S_i = 0$ could be traced back to non-trivial effects of orientifolds in the string theory realizations [25].

the first n S_i's non-vanishing, and the remaining $m-n$ as zero. From this we can infer the "dual" effective superpotential following [7]. For pseudo-confining vacua, and in the classical limit, we have

$$\begin{aligned} W_{\text{eff}} = & -\frac{1}{2}\sum_{i=1}^{n} N_i \int_{\widehat{B}_i^r} y_{m.m.}(z)dz - \frac{1}{2}\sum_{f=1}^{N_f}\int_{-m_f}^{\widetilde{\Lambda}_0} y_{m.m.}(z)dz + \\ & +\frac{1}{2}(2N-N_f)W(\Lambda_0) + \frac{1}{2}\sum_{f=1}^{N_f} W(-m_f) - \\ & -\pi i(2N-N_f)S + S\log\left(\frac{\Lambda^{2N-N_f}}{\Lambda_0^{2N-N_f}}\right) + 2\pi i\sum_{i=1}^{n-1} c_i S_i\,. \end{aligned} \tag{3.23}$$

where $\widehat{B}_i$ are compact cycles that together with the first $n-1$ A_i cycles form a symplectic basis, c_i are the (integer-) periods of the T differential on these cycles and Λ_0 is a cut-off. The classical limit has been computed in [7] and in the pseudo-confining branch it was shown to be

$$W_{\text{eff}}\big|_{\Lambda\to 0} = \sum_{i=1}^{n} N_i\, W(\varphi_i)\,. \tag{3.24}$$

The derivative of the effective superpotential with respect to the logarithm of the scale is

$$\frac{\partial W_{\text{eff}}}{\partial \log \Lambda^{b_0}} = \sum_{i=1}^{m} S_i = S = \frac{b_{m-1}}{4g_m}\,. \tag{3.25}$$

We used the fact that S can be computed easily from the matrix model curve as a period integral over a large circle of $S = \frac{1}{2}\oint_{\mathcal{C}_{\text{large}}} y_{m.m.}(z)\,dz$ [17]. Hence we can obtain the value of the derivative

$$\frac{\partial W_{\text{eff}}(\langle S_i\rangle\,,\Lambda)}{\partial \log \Lambda^{b_0}} = S = \frac{b_{m-1}}{4g_m}\,, \tag{3.26}$$

where in the last identity we used (3.25).

This shows the equivalence between the field theory and the "dual" geometry effective superpotentials. The argument as it has been presented is valid only for the pseudo-confining vacua. In the quantum theory the Higgs vacua are continuously connected to them by varying the masses m_f in such a way that they cross a branch cut of the matrix model curve. We can compute the superpotential in an arbitrary vacuum by first computing it on the pseudo-confining branch and the analytically continuing the values of m_f. In this way the equivalence extends to the pseudo-higgs branches as well.

3.3 Examples

In this subsection we discuss a couple of examples to show how the duality claimed in subsection 2.1 works. We will consider first the example of a U(2) theory with no flavors and cubic superpotential. We trivially find the same Riemann surface for both descriptions and thus the dual theory is the same as the original one.

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As a second example we will consider the case of a U(3) gauge group without fundamental matter and again a cubic tree level superpotential. This is how we regard the traditional picture of the M-theory curve, therefore obtaining the respective results in the dual interpretation.

$P_N(v)$ is given in terms of the symmetric polynomials s_i as

$$P_N(v) = \det(v - \phi) = v^N + s_1 v^{N-1} + \cdots + s_N\,, \tag{3.27}$$

The factorization conditions determine the vacuum expectation values of the s_i's and from (3.3) we find the vevs of the Casimirs U_k. These values have to be simply plugged into the tree-level superpotential to compute $W_{\rm low}$.

'U(2) gauge theory, no matter'. Starting with the traditional picture (this is a choice), we have $N = 2$, $N_f = 0$ and $\deg W' = 2$. Since $P_N = v^2 + s_1 v + s_2$, the SW curve is

$$\mathrm{SW}: \quad (v^2 + s_1 v + s_2)^2 - 4\Lambda^4\,, \tag{3.28}$$

whereas we take $W_{\rm tree}$ as

$$W_{\rm tree} = U_3 + mU_2 + \ell U_1 \quad \rightarrow \quad W_{\rm tree}(v) = \frac{1}{3}v^3 + \frac{m}{2}v^2 + \ell v\,, \tag{3.29}$$

so the DV curve is

$$\mathrm{DV}: \quad (v^2 + mv + \ell)^2 - (f_1 v + f_0)\,. \tag{3.30}$$

The idea now is to solve for $\{s_1, s_2, f_0, f_1\}$ in terms of the quantities of the theory by using factorization of the curves. The trivial case corresponds to the choice $n = 2$ (could be $n = 1$ too.) Then both $S(v)$ and $H(v)$ are constants which we can set to be equal. Hence, the two curves are automatically equal so

$$(v^2 + s_1 v + s_2)^2 - 4\Lambda^4 = (v^2 + mv + \ell)^2 - (f_1 v + f_0)\,, \tag{3.31}$$

from where one obtains

$$s_1 = m\,, \qquad s_2 = \ell\,, \qquad f_0 = 4\Lambda^4\,, \qquad f_1 = 0\,. \tag{3.32}$$

Now, for this theory the U_k's are given by

$$U_1 = -s_1\,, \qquad U_2 = \frac{U_1^2}{2} - s_2\,, \qquad U_3 = U_1\,U_2 - \frac{U_1^3}{6}\,, \tag{3.33}$$

so once we found the values for s_1, s_2 we use (3.33) and (3.32) to find

$$U_1 = -m\,, \qquad U_2 = \frac{m^2}{2} - \ell\,, \qquad U_3 = m\ell - \frac{m^3}{3}\,, \tag{3.34}$$

$$W_{\rm low} = \frac{m^3}{6} - m\ell\,. \tag{3.35}$$

As said, this is just one way of regarding the projections of the M-theory curve. The alternative (dual) one is to do the change SW $\rightarrow \widetilde{\mathrm{DV}}$ and DV $\rightarrow \widetilde{\mathrm{SW}}$. However, in this trivial example the new theory we obtain is also a U(2) gauge theory without matter (since $f_1 = 0$). Because the factorization problem is the same (this is always true), the low energy effective superpotential has the same shape in this picture

$$W_{\rm low}^{dual} = \frac{m^3}{6} - m\ell\,. \tag{3.36}$$

'U(3) gauge theory, no matter'. Here we consider a somewhat more interesting case. We will regard the traditional picture as that with $N = 3$, $N_f = 0$ and $\deg W' = 2$. The curves are

$$\text{SW}: \ (v^3 + s_1 v^2 + s_2 v + s_3)^2 - 4\Lambda^6 \,, \tag{3.37}$$

$$\text{DV}: \ (v^2 + mv + \ell)^2 - (f_1 v + f_0) \,. \tag{3.38}$$

For the choice of $n = 2$, $H(v)$ is a constant. The factorization problem thus translates into

$$(v^3 + s_1 v^2 + s_2 v + s_3)^2 - 4\Lambda^6 = (v-a)^2 \left[(v^2 + mv + \ell)^2 - (f_1 v + f_0)\right] , \tag{3.39}$$

having four different solutions. Defining $\Delta^2 \equiv m^2 - 4\ell$, they are reached through

$$\Lambda^3 \to -\Lambda^3 \,, \qquad \Delta \to -\Delta \,.$$

To be concise we will consider the solution with plus signs for Λ^3 and Δ

$$\begin{aligned} s_1 &= \frac{3m - \Delta}{2} \,, & s_2 &= \ell + \frac{m(m-\Delta)}{2} \,, & s_3 &= \frac{\ell(m-\Delta)}{2} + 2\Lambda^3 \\ f_0 &= -2\Lambda^3 (m + \Delta) \,, & f_1 &= -4\Lambda^3 \,, & a &= \frac{\Delta - m}{2} \,. \end{aligned} \tag{3.40}$$

From these one can obtain the U_k's

$$U_1 = \frac{\Delta - 3m}{2} \,, \qquad U_2 = \frac{m(3m - \Delta)}{4} - \frac{3\ell}{2} \,, \qquad U_3 = \frac{3m\ell}{2} + \frac{\Delta(m^2 - \ell)}{6} - \frac{m^3}{2} - 2\Lambda^3 \,, \tag{3.41}$$

which allows us to write the effective superpotential

$$W_{\text{low}} = \frac{m^3}{4} - \frac{3m\ell}{2} - \frac{\Delta^3}{12} - 2\Lambda^3 \,. \tag{3.42}$$

Let us now go to the dual picture and see what the superpotential is in terms of other variables. From the previous calculations we already know all of the quantities. The curves are

$$\widetilde{\text{SW}}: \ (v^2 + mv + \ell)^2 - (f_1 v + f_0) \,, \tag{3.43}$$

$$\widetilde{\text{DV}}: \ (v^3 + s_1 v^2 + s_2 v + s_3)^2 - 4\Lambda^6 \,. \tag{3.44}$$

so then we know

$$\begin{cases} \widetilde{s}_1 = m \,, \quad \widetilde{s}_2 = \ell \\ \widetilde{F}_0 = f_0 \,, \quad \widetilde{F}_1 = f_1 \\ \text{Couplings of superpotential: } \{s_1, s_2, s_3\} \end{cases} \tag{3.45}$$

As one can see, for this dual theory we have matter and a *quartic* superpotential with different couplings, which we write as

$$\widetilde{W}_{\text{tree}} = \widetilde{U}_4 + s_1 \widetilde{U}_3 + s_2 \widetilde{U}_2 + s_3 \widetilde{U}_1 \,. \tag{3.46}$$

There is one dual flavor with mass $\widetilde{m} = (m+\Delta)/2$ and the scale of the dual theory is $\widetilde{\Lambda}^3 = \Lambda^3$! Since in this case matter is present the relations between the s_i's and the U_k's suffer a modification

$$\widetilde{U}_1 = -\widetilde{s}_1\,, \qquad \widetilde{U}_2 = \frac{\widetilde{U}_1^2}{2} - \widetilde{s}_2\,, \qquad \widetilde{U}_3 = \widetilde{U}_1\,\widetilde{U}_2 - \frac{\widetilde{U}_1^3}{6} + \frac{\widetilde{F}_1}{4} \tag{3.47}$$

$$\widetilde{U}_4 = \widetilde{U}_1\,\widetilde{U}_3 + \frac{\widetilde{U}_2^2}{2} + \frac{\widetilde{U}_1^4}{24} - \frac{\widetilde{U}_1^2\,\widetilde{U}_2}{2} + \frac{\widetilde{F}_1\widetilde{U}_1}{4} + \frac{\widetilde{F}_0}{4}\,, \tag{3.48}$$

so plugging the expressions we know for $\{\widetilde{s}_1, \widetilde{s}_2, \widetilde{F}_0, \widetilde{F}_1\}$ yields

$$\widetilde{U}_1 = -m\,, \qquad \widetilde{U}_3 = m\ell - \frac{m^3}{3} + \Lambda^3 \tag{3.49}$$

$$\widetilde{U}_2 = \frac{m^2}{2} - \ell\,, \qquad \widetilde{U}_4 = \frac{m^4}{4} + \frac{\ell^2}{2} - \frac{3m+\Delta}{2}\Lambda^3 - m^2\ell\,, \tag{3.50}$$

while further plugging in the values of $\{s_1, s_2, s_3\}$ (which are interpreted as couplings now) finally gives the effective superpotential for this dual interpretation

$$\frac{1}{g_3} W_{\rm low}^{dual} = \frac{m\ell\Delta}{6} - \frac{m\Delta^3}{12} - \frac{\ell^2}{2} - 2m\Lambda^3\,. \tag{3.51}$$

See the results obtained are different for these two theories, even though they come from the same M-theory curve.

4. Discussion and conclusions

We have seen that the M-theory curve describing $\mathcal{N}=1$ gauge theories allows two dual interpretations upon exchanging what one views as the Seiberg-Witten or as the Dijkgraaf-Vafa curve. There are two gauge theories which give rise to the same curve embedded in a three dimensional complex space. The dual interpretations are possible only for the ground-state since the fluctuations in the two systems are different. However the factorization conditions contain enough information to compute the low energy effective superpotential in both theories.

The duality exchanges the rank of the gauge group N with the degree of the superpotential $m+1$ and furthermore the deformations of the Dijkgraaf-Vafa curve with the mass terms of the Seiberg-Witten curve. Because of the exchange of m and N one always has to deal with a tree level superpotential of high degree, i.e. operators $\mathrm{tr}(\phi)$ with $k > N$. We therefore had to extend the known proofs in the literature of the equivalence of the field theory effective superpotential with the DV (or matrix model) superpotential to cases of arbitrary k and a number of flavors $N_f < 2N$. Another intriguing aspect of the duality is that the constraint of allowing at most log-normalizable deformations of the DV curve is mapped to a constraint in the number of flavors $N_f < N$. This is rather interesting since it is well known that the low energy physics of gauge theories with more flavors, although being qualitatively different, is still well defined. So one might ask then what is the significance of the deformations exceeding the log-normalizable bound in the DV curve. We leave this question for future investigation.

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